Course Material

Ordinary Differential Equations

Differential Equation:

A differential equation is an equation involving differential or differential coefficients.

Ordinary differential equations: Differential equations which involve only one independent variable and differential co-efficients w.r.t it are called Ordinary differential equations.

Partial differential equations: Differential equations which involve two or more independent variable and partial derivatives w.r.t it are called Partial differential equations.

Exact Differential Equation: A differential equation is called exact differential equation if it is obtained only by differentiation of its primitive without applying any other operation like addition, subtraction, elimination etc.

e.g. $xdy + ydx = 0$

Necessary and Sufficient condition for differential eq. to be exact:
The necessary and sufficient condition for differential eq.

Mdx+Ndy =0 to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Solution of Exact Differential equation:

$$\int_{y\ constant} M\,dx \quad + \int terms\ of\ N\ free\ from\ x\ dy = c$$

Integrating Factor: A suitable factor by multiplying with which, the given equation becomes exact is called integrating factor.

Rules to find Integrating Factor:

1. By Inspection:
   i. $xdy + ydx = d(xy)$
   ii. $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$
   iii. $\frac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$
   iv. $\frac{ydx - xdy}{x^2 + y^2} = d\left(\log\frac{x}{y}\right)$
   v. $\frac{ydx - xdy}{xy} = d(\log xy)$

2. If the equation Mdx+Ndy =0 is homogenous, then
   Integrating factor $= \frac{1}{Mx + Ny}$, provided $Mx + Ny$
3. If the equation Mdx+Ndy =0 is of the form $f(xy)ydx + g(xy)xdy = 0$, then
   Integrating factor $= \frac{1}{Mx - Ny}$, provided $Mx - Ny$
4. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x only, then $\text{I.F.} = e^{\int f(x)dx}$

5. If $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = g(y)$, a function of y only, then $\text{I.F.} = e^{\int g(y)dy}$
6. If equation Mdx+Ndy =0 is of the form
   $x^a y^b (mydx + nxdy) + x^c y^d (pydx + qxdy) = 0$,
   a, b, c, d, m, n, p, q are constants, then $\text{I.F.} = x^h y^k$, where h and k are obtained from the equations
$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \quad and \quad \frac{c+h+1}{p} = \frac{d+k+1}{q}$$

Equations solvable for p $\left(p = \frac{dy}{dx}\right)$

- Let the differential equation be of degree n.
- Factorize the differential equation into the linear factors of the form $(p - f(x,y))$
- Each factor is a differential equation of first order and first degree. Solve them.
- Let the solutions of these factors be
$$g_1(x,y,c) = 0, g_2(x,y,c) = 0, \quad g_3(x,y,c) = 0, \dots\dots\dots\dots\dots, \quad g_n(x,y,c) = 0$$
- Then solution of given equation is given by
$$g_1(x,y,c)g_2(x,y,c)g_3(x,y,c) \dots\dots\dots\dots\dots g_n(x,y,c) = 0$$

Equations solvable for y

- Write the differential equation in the form $y = f(x,p) \dots\dots\dots\dots\dots (1)$
- Differentiating both sides w.r.t. x, we get $p = F\left(x, p, \frac{dp}{dx}\right) \dots\dots\dots\dots\dots (2)$
- (2) is a differential equation of first order and first degree in $\frac{dp}{dx}$. Solve it.
- Let the solution of (2) be $g(x,p,c) = 0 \dots\dots\dots\dots\dots (3)$
- To obtain the solution of given equation, eliminate p from (1) and (3).
- If it is not possible to eliminate p from (1) and (3), then using (1) and (3), find the values of y and x in terms of p and c. let these values be
$$y = \emptyset(p,c) \dots\dots\dots\dots\dots (4) \quad and \quad x = \psi(x,p) \dots\dots\dots\dots\dots (5)$$
- (4) and (5) simultaneously give solution of given equation.

Equations solvable for x

- Write the differential equation in the form $x = f(y,p) \dots\dots\dots\dots\dots (1)$
- Differentiating both sides w.r.t. y, we get $\frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right) \dots\dots\dots\dots\dots (2)$
- (2) is a differential equation of first order and first degree in $\frac{dp}{dy}$. Solve it.
- Let the solution of (2) be $g(y,p,c) = 0 \dots\dots\dots\dots\dots (3)$
- To obtain the solution of given equation, eliminate p from (1) and (3).
- If it is not possible to eliminate p from (1) and (3), then using (1) and (3), find the values of y and x in terms of p and c. let these values be
$$y = \emptyset(p,c) \dots\dots\dots\dots\dots (4) \quad and \quad x = \psi(x,p) \dots\dots\dots\dots\dots (5)$$
- (4) and (5) simultaneously give solution of given equation.

Clairaut's Equation: An equation of the form $y = px + f(p)$ is called Clairaut's equation. Its solution is given by replacing p by c in the given equation.

Equations reducible to Clairaut's equation:

i. If the equation contains terms of the type e^{ax} and e^{by},
   Put $e^{hx} = X$ and $e^{hy} = Y$, where $h = H.C.F.(a, b)$.

ii. If the equation is of the form $y = \left(\frac{py}{x}\right)x^2 + f\left(\frac{py}{x}\right)$
   Put $x^2 = X$ and $y^2 = Y$.

Formation of a differential equation:-

Q.1 Eliminate the constants from the equation

$y = e^x(A\cos x + B\sin x)$ and obtain the differential equation.

Sol. $y = e^x(A\cos x + B\sin x)$ ……………(1)

Diff.equation (1) w.r.t x

$$\frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) \qquad \text{………………(2)}$$

Again diff.equation (2) w.r.t x

$$\frac{d^2y}{dx^2} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x) \ (\because \text{fromequation(2)})$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + (\frac{dy}{dx} - y) - y$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Assignment:

Eliminate the arbitrary constants and the differential equations:-

(1) $y = cx + c^2$

(2) $y = Ae^x + Be^{-x} + C$

(3) $xy = Ae^x + Be^{-x} + x^2$

(4) Find the differential equation of all parabolas whose axes are parallel to y-axis.

(5) Form the differential equation of all circles of radius a.

Solution of differential equations of the first order and first degree:-

1. Variables Separable Form
2. Homogeneous equation
3. Equations Reducible to Homogeneous form
4. Exact differential equation

Variables Separable Form:-

If a differential equation of the first order and first degree

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)dy = f(x)dx$$

$$\int f(x)dx = \int g(y)dy$$

Q2. Solve $x\cos x\cos y + \sin y\frac{dy}{dx} = 0$

Sol. $x\cos x\cos y + \sin y\frac{dy}{dx} = 0$

$$x\cos x\cos y = -\sin y\frac{dy}{dx}$$

$$x\cos x = -\tan y\frac{dy}{dx}$$

$$x\cos xdx = -\tan ydy$$

$$\int \underset{1}{x}\underset{11}{\cos x}\,dx = -\int \tan ydy$$

$$x\sin x - \int 1.\sin xdx = \log\cos y$$

$$x\sin x + \cos x = \log\cos y + c$$

Assignment:-

Solve the following differential equations:-

(1) $\frac{dy}{dx} = 1 + x + y + xy$

(2) $(x + y + 1)^2 \frac{dy}{dx} = 1$

(3) $xy \frac{dy}{dx} = 1 + x + y + xy$

(4) $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

(5) $\frac{dy}{dx} = \cos(x + y)$

Homogeneous equation:-

If a differential equation of the first order and first degree

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Putting y =vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation(1)

$$v + x\frac{dv}{dx} = f\left(\frac{vx}{x}\right)$$

$$v + x\frac{dv}{dx} = f(v)$$

$$f(v) - v = x\frac{dv}{dx}$$

$$\frac{dv}{(f(v) - v)} = \frac{dx}{x}$$

$$\int \frac{dv}{(f(v) - v)} = \int \frac{dx}{x} + C$$

Q.3 Solve $xdy - ydx = \sqrt{x^2 + y^2}\,dx$

Sol. $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \quad \ldots\ldots\ldots\ldots\ldots(1)$

Putting y =vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation(1)

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

$$\int\frac{dv}{\sqrt{1+v^2}} = \int\frac{dx}{x}$$

$$\log(v + \sqrt{1+v^2}) = \log(x) + \log c$$

$$v + \sqrt{1+v^2} = cx$$

$$y + \sqrt{x^2 + y^2} = cx^2$$

Assignment:-

Solve the following differential equations:-

(1) $(x\tan\frac{y}{x} - y\sec^2\frac{y}{x})dx + x\sec^2\frac{y}{x}dy = 0$

(2) $(x+y)dx + (x-y)dy = 0$

(3) $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$

(4) $(1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}(1 - \frac{x}{y})dy = 0$

(5) $ye^{\frac{x}{y}}dx = (xe^{\frac{x}{y}} + y)dy$

Equations Reducible to Homogeneous form:-

A differential equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{a^{'}x + b^{'}y + c^{'}}$ can be reduced to homogeneous form

When $\frac{a}{a^{'}} \neq \frac{b}{b^{'}}$

Putting x=X+h ,y=Y+k

dx=dX,dy=dY

$$\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{a^{'}(X+h)+b^{'}(Y+k)+c^{'}}$$

$$\frac{dY}{dX}=\frac{aX+bY+ah+bk+c}{a^{'}X+b^{'}Y+a^{'}h+b^{'}k+c^{'}} \qquad \ldots\ldots\ldots\ldots(1)$$

$$ah+bk+c=0$$

$$a^{'}h+b^{'}k+c^{'}=0$$

$$\frac{h}{bc^{'}-b^{'}c}=\frac{k}{ca^{'}-c^{'}a}=\frac{1}{ab^{'}-a^{'}b}$$

$$h=\frac{bc^{'}-b^{'}c}{ab^{'}-a^{'}b} \quad ,k=\frac{ca^{'}-c^{'}a}{ab^{'}-a^{'}b}$$

From equation (1)

$$\frac{dY}{dX}=\frac{aX+bY}{a^{'}X+b^{'}Y} \quad \text{which is homogenous in } X,Y \text{ and can be solved by}$$

$$\text{putting} \quad Y=vX$$

State and Prove Necessary and sufficient condition for differential eq. Mdx+Ndy =0 to be exact.

Sol.<u>Statement:</u> The necessary and sufficient condition for differential eq. Mdx+Ndy =0 to be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Proof: The equation M dx + N dy = 0 will be exact if

$$du=Mdx+Ndy$$

But $\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=du$

$$\therefore Mdx+Ndy=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

Equating coefficients of dx and dy, we get

$$M=\frac{\partial u}{\partial x} \quad and \quad Mdx+N=\frac{\partial u}{\partial y}\xrightarrow{yields}\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x} \quad and \quad \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}$$

$$But \quad \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}\xrightarrow{yields}\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

- Solve $(5x^4+3x^2y^2-2xy^3)dx+(2x^3y-3x^2y^2-5y^4)dy=0$

Sol. Here $M=(5x^4+3x^2y^2-2xy^3)$ and $N=2x^3y-3x^2y^2-5y^4)$

$\frac{\partial M}{\partial y}=6x^2y-6xy^2$ and $\frac{\partial N}{\partial x}=6x^2y-6xy^2$

Therefore $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx+\int terms\ of\ N\ free\ from\ x\ dy=c$

$$\xrightarrow{yields}\int(5x^4+3x^2y^2-2xy^3)dx+\int(-5y^4)dy=c \xrightarrow{yields} x^5+x^3y^2-x^2y^3-y^5=c$$

- Check whether the given equation $(1+x^2)dy + 2xydx = 0$ is exact and obtain the general solution .

Sol.Here M=2xy and N=$(1+x^2)$

$\frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int 2xydx + \int 1.dy = c \xrightarrow{yields} x^2y + y = c$$

- Find the value of a so that the differential equation $xy^3dx + ax^2y^2dy = 0$ is exact.

Sol.Here M= $xy^3$and N=ax^2y^2

$\frac{\partial M}{\partial y} = 3xy^2$ and $\frac{\partial N}{\partial x} = 2axy^2$

Hence equation is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \xrightarrow{yields} 3xy^2 = 2axy^2 \xrightarrow{yields} a = \frac{3}{2}$

- Solve e^x(cos y dx-sin y dy)=0 ; y(0)=0

Sol. Here $M = e^x \cos y$and $N = -e^x \sin y$

$\frac{\partial M}{\partial y} = -e^x \sin y$ and $\frac{\partial N}{\partial x} = -e^x \sin y$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int e^x \cos y\, dx + \int (0)dy = c \xrightarrow{yields} e^x \cos y = c$$

- Solve $(x^2-ay)dx=(ax-y^2)dy$

Sol. Here $M = x^2 - ay$ and $N = y^2 - ax$

$\frac{\partial M}{\partial y} = -a$ and $\frac{\partial N}{\partial x} = -a$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int (x^2 - ay)dx + \int (y^2)dy = c \xrightarrow{yields} \frac{x^3}{3} - axy + \frac{y^3}{3} = c$$

- Solve (sec x tan x tan y-e^x)dx+sec x $\sec^2$ydy=0

Sol. Here $M = secxtanxtany - e^x$and $N = sec\,x\,sec^2 y$

$\frac{\partial M}{\partial y} = secxtanxsec^2y$ and $\frac{\partial N}{\partial x} = secxtanxsec^2y$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int (secxtanxtany - e^x)dx + \int (0)dy = c \xrightarrow{yields} secxtany - e^x = c$$

- Solve $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$

Sol. Here $M = x^2 - 4xy - 2y^2$and $N = y^2 - 4xy - 2x^2$

$\frac{\partial M}{\partial y} = -4x - 4y$ and $\frac{\partial N}{\partial x} = -4y - 4x$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int (x^2 - 4xy - 2y^2)dx + \int (y^2)dy = c \xrightarrow{yields} \frac{x^3}{3} - 2x^2y - 2y^2x + \frac{y^3}{3} = c$$

- Solve $(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$

Sol. Here $M = 3x^2 + 6xy^2$ and $N = 6x^2y + 4y^3$

$\frac{\partial M}{\partial y} = 12xy$ and $\frac{\partial N}{\partial x} = 12xy$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int (3x^2 + 6xy^2)dx + \int (4y^3)dy = c \xrightarrow{yields} x^3 + 3x^2y^2 + y^4 = c$$

- Solve $(x^2+y^2-a^2)x\,dx+(x^2-y^2-b^2)y\,dy=0$

Sol. Here $M = x^3 + y^2x - xa^2$ and $N = x^2y - y^3 - y\,b^2$

$\frac{\partial M}{\partial y} = 2xy$ and $\frac{\partial N}{\partial x} = 2xy$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int (x^3 + y^2x - xa^2)dx + \int (-y^3 - y\,b^2)dy = c \xrightarrow{yields} \frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{a^2x^2}{2} - \frac{y^4}{4} - \frac{y^2b^2}{2} = c$$

- Solve xdy+ydx+ $\left(\frac{xdy - ydx}{x^2 + y^2}\right) = 0$

Sol. Here $M = y - \frac{y}{x^2+y^2}$ and $N = x + \frac{x}{x^2+y^2}$

$\frac{\partial M}{\partial y} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}$ and $\frac{\partial N}{\partial x} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int \left(y - \frac{y}{x^2 + y^2}\right) dx + \int (0)dy = c \xrightarrow{yields} yx - tan^{-1}\frac{x}{y} = c$$

- Solve $ydx-xdy+3x^2y^2 e^{x^3}$ dx =0

Sol. Dividing both sides by y^2, we get $\frac{ydx-xdy}{y^2} + 3x^2e^{x^3}dx = 0 \xrightarrow{yields} d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0$

Integrating both sides, we get $\frac{x}{y} + e^{x^3} = c$

- Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

Sol. Here $M = x^2y - 2xy^2$ and $N = 3x^2y - x^3$

$\frac{\partial M}{\partial y} = x^2 - 4xy$ and $\frac{\partial N}{\partial x} = 6xy - 3x^2$

Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence equation is not exact.

Since the given equation is homogenous, so I.F. = $\frac{1}{Mx+Ny} = \frac{1}{x^3y-2x^2y^2+3x^2y^2-x^3y} = \frac{1}{x^2y^2}$

Multiplying the given equation by $\frac{1}{x^2y^2}$, we get $\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$

Now $M=\left(\frac{1}{y}-\frac{2}{x}\right)$and $N=-\left(\frac{x}{y^2}-\frac{3}{y}\right)$

$\frac{\partial M}{\partial y}=\frac{-1}{y^2}$and $\frac{\partial N}{\partial x}=\frac{-1}{y^2}$

Therefore $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx+\int terms\ of\ N\ free\ from\ x\ dy=c$

$$\xrightarrow{yields}\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\left(\frac{3}{y}\right)dy=c\xrightarrow{yields}\frac{x}{y}-2\log x+3\log y=c$$

- Solve xdy-ydx=(x^2+y^2)dx

Sol. Dividing both sides by x^2+y^2, we get $\frac{xdy-ydx}{x^2+y^2}=dx\xrightarrow{yields}d\left(tan^{-1}\frac{y}{x}\right)=dx\xrightarrow{yields}tan^{-1}\frac{y}{x}-x=c$

- Solve xdy-ydx=xy^2 dx

Sol. Dividing both sides by y^2, we get $\frac{xdy-ydx}{y^2}=xdx\xrightarrow{yields}d\left(\frac{-x}{y}\right)=xdx\xrightarrow{yields}\frac{-x}{y}=\frac{x^2}{2}+c$

- Solve (1+xy)ydx+(1-xy)xdy=0

Sol. Here $M=y+xy^2$and $N=x-x^2y$

$\frac{\partial M}{\partial y}=1+2xy$ and $\frac{\partial N}{\partial x}=1-2xy$

Therefore $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ Hence equation is not exact.

Since the given equation is of the form $(xy)ydx+g(xy)xdy=0$, I.F. = $\frac{1}{Mx-Ny}=\frac{1}{2x^2y^2}$

Multiplying the given equation by$\frac{1}{2x^2y^2}$, we get $\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\left(\frac{1}{2xy^2}-\frac{1}{2y}\right)dy=0$

Now $M=\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)$and $N=\left(\frac{1}{2xy^2}-\frac{1}{2y}\right)$

$\frac{\partial M}{\partial y}=\frac{-1}{2x^2y^2}$and $\frac{\partial N}{\partial x}=\frac{-1}{2x^2y^2}$

Therefore $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx+\int terms\ of\ N\ free\ from\ x\ dy=c$

$$\xrightarrow{yields}\int\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\int\left(-\frac{1}{2y}\right)dy=c\xrightarrow{yields}\frac{-1}{2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$$

- Solve x^2ydx-(x^3+y^3)dy=0

Sol. Here $M=x^2y$ and $N=-x^3-y^3$

$\frac{\partial M}{\partial y}=x^2$and $\frac{\partial N}{\partial x}=-3x^2$

Therefore $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ Hence equation is not exact.

Since the given equation is homogenous, so I.F. = $\frac{1}{Mx+Ny}=\frac{1}{x^3y-x^3y-y^4}=\frac{-1}{y^4}$

Multiplying the given equation by$\frac{-1}{y^4}$, we get $\left(\frac{-x^2}{y^3}\right)dx+\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=0$

Now $M=\left(\frac{-x^2}{y^3}\right)$and $N=\left(\frac{x^3}{y^4}+\frac{1}{y}\right)$

$\frac{\partial M}{\partial y} = \frac{3x^2}{y^4}$and $\quad \frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int \left(\frac{-x^2}{y^3}\right)dx + \int \left(\frac{1}{y}\right)dy = c \xrightarrow{yields} \frac{x^3}{3y^3} + \log y = c$$

- Solve $(2x^2y-3y^4)dx+(3x^3+2xy^3)dy=0$

Sol. Here $M = 2x^2y - 3y^4$and $\quad N = 3x^3 + 2xy^3$

$\frac{\partial M}{\partial y} = 2x^2 - 12y^3$and $\quad \frac{\partial N}{\partial x} = 9x^2 + 2y^3$

Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence equation is not exact.

Now equation is of the form

$x^a y^b (mydx + nxdy) + x^c y^d (pydx + qxdy) = 0$,

$$\therefore \text{I. F.} = x^a y^b = x^{\frac{-28}{13}} y^{\frac{-49}{13}}$$

Multiplying the given equation by $x^{\frac{-49}{13}} y^{\frac{-28}{13}}$, we get

$$\left(2x^{\frac{-23}{13}} y^{\frac{-15}{13}} - 3x^{\frac{-49}{13}} y^{\frac{24}{13}}\right)dx + \left(3x^{\frac{-10}{13}} y^{\frac{-28}{13}} + 2x^{\frac{-36}{13}} y^{\frac{-11}{13}}\right)dy = 0$$

Now $M = \left(2x^{\frac{-23}{13}} y^{\frac{-15}{13}} - 3x^{\frac{-49}{13}} y^{\frac{24}{13}}\right)$

and $\quad N = \left(3x^{\frac{-10}{13}} y^{\frac{-28}{13}} + 2x^{\frac{-36}{13}} y^{\frac{11}{13}}\right)$

$\frac{\partial M}{\partial y} = \frac{-30}{13} x^{\frac{-23}{13}} y^{\frac{-28}{13}} - \frac{72}{13} x^{\frac{-49}{13}} y^{\frac{11}{13}}$and $\quad \frac{\partial N}{\partial x} = \frac{-30}{13} x^{\frac{-23}{13}} y^{\frac{-28}{13}} - \frac{72}{13} x^{\frac{-49}{13}} y^{\frac{11}{13}}$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int \left(2x^{\frac{-23}{13}} y^{\frac{-15}{13}} - 3x^{\frac{-49}{13}} y^{\frac{24}{13}}\right)dx + \int 0dy = c \xrightarrow{yields} \frac{-13}{5} x^{\frac{-10}{13}} y^{\frac{-15}{13}} + \frac{13}{12} x^{\frac{-36}{13}} y^{\frac{24}{13}} = c$$

- Solve $(3x^2y^3e^y + y^3 + y^2)dx + (x^3y^3e^y - xy)dy = 0$

Sol. Here $M = (3x^2y^3e^y + y^3 + y^2)$and $\quad N = (x^3y^3e^y - xy)$

$\frac{\partial M}{\partial y} = (9x^2y^2e^y + 3x^2y^3e^y + 3y^2 + 2y)$and $\quad \frac{\partial N}{\partial x} = 3x^2y^3e^y - y$

Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence equation is not exact.

Now $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(3x^2y^3e^y - y) - (9x^2y^2e^y + 3x^2y^3e^y + 3y^2 + 2y)}{3x^2y^3e^y + y^3 + y^2} = \frac{-3}{y} = g(y)$, a function of y only.

$$\therefore \text{I. F.} = e^{\int g(y)dy} = e^{\int \frac{-3}{y}dy} = e^{-3\ln y} = \frac{1}{y^3}$$

Multiplying the given equation by$\frac{1}{y^3}$, we get

$$\left(3x^2e^y + 1 + \frac{1}{y}\right)dx + \left(x^3e^y - \frac{x}{y^2}\right)dy = 0$$

Now $M = \left(3x^2e^y + 1 + \frac{1}{y}\right)$

and $N = \left(x^3e^y - \frac{x}{y^2}\right)$

$\frac{\partial M}{\partial y} = \left(3x^2e^y - \frac{1}{y^2}\right)$and $\frac{\partial N}{\partial x} = \left(3x^2e^y - \frac{1}{y^2}\right)$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int \left(3x^2e^y + 1 + \frac{1}{y}\right)dx + \int 0dy = c \xrightarrow{yields} x^3e^y + x + \frac{x}{y} = c$$

- Solve $(5x^3 + 12x^2 + 6y^2)dx + 6xydy = 0$

Sol. Here $M = (5x^3 + 12x^2 + 6y^2)$and $N = 6xy$

$\frac{\partial M}{\partial y} = 12y$and $\frac{\partial N}{\partial x} = 6y$

Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence equation is not exact.

Now $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{12y - 6y}{6xy} = \frac{1}{x} = f(x)$, a function of x only.

$$\therefore \text{I.F.} = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Multiplying the given equation by$x$, we get

$$(5x^4 + 12x^3 + 6xy^2)dx + 6x^2ydy = 0$$

Now $M = (5x^4 + 12x^3 + 6xy^2)$

and $N = 6x^2y$

$\frac{\partial M}{\partial y} = 12xy$and $\frac{\partial N}{\partial x} = 12xy$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int (5x^4 + 12x^3 + 6xy^2)dx + \int 0dy = c \xrightarrow{yields} x^5 + 3x^4 + 3x^2y^2 = c$$

- Solve $(2x^2y^2 + y)dx + (3x - x^3y)dy$

Sol. Here $M = (2x^2y^2 + y)$ and $N = (3x - x^3y)$

$\frac{\partial M}{\partial y} = 4x^2y + 1$ and $\frac{\partial N}{\partial x} = 3 - 3x^2y$

Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence equation is not exact.

Given Equation can be written as

$x^2y(2ydx - xdy) + x^0y^0(ydx + 3xdy) = 0$

Now equation is of the form

$x^ay^b(mydx + nxdy) + x^cy^d(pydx + qxdy) = 0$,

$$\therefore \text{I.F.} = x^hy^k = x^{\frac{-11}{7}}y^{\frac{-19}{7}}$$

Multiplying the given equation by $x^{\frac{-11}{7}}y^{\frac{-19}{7}}$, we get

$$\left(2x^{\frac{3}{7}}y^{\frac{-5}{7}} + x^{\frac{-11}{7}}y^{\frac{-12}{7}}\right)dx + \left(3x^{\frac{-4}{7}}y^{\frac{-19}{7}} - x^{\frac{10}{7}}y^{\frac{-12}{7}}\right)dy = 0$$

Now $M = \left(2x^{\frac{3}{7}}y^{\frac{-5}{7}} + x^{\frac{-11}{7}}y^{\frac{-12}{7}}\right)$

and $N = \left(3x^{\frac{-4}{7}}y^{\frac{-19}{7}} - x^{\frac{10}{7}}y^{\frac{-12}{7}}\right)$

$\frac{\partial M}{\partial y} = \frac{-10}{7}x^{\frac{3}{7}}y^{\frac{-12}{7}} - \frac{12}{7}x^{\frac{-11}{7}}y^{\frac{-19}{7}}$ and $\frac{\partial N}{\partial x} = \frac{-10}{7}x^{\frac{3}{7}}y^{\frac{-12}{7}} - \frac{12}{7}x^{\frac{-11}{7}}y^{\frac{-19}{7}}$ Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$\xrightarrow{yields} \int \left(2x^{\frac{3}{7}}y^{\frac{-5}{7}} + x^{\frac{-11}{7}}y^{\frac{-12}{7}}\right)dx + \int 0dy = c$

$\xrightarrow{yields} \frac{7}{5}x^{\frac{10}{7}}y^{\frac{-5}{7}} - \frac{7}{4}x^{\frac{-4}{7}}y^{\frac{-12}{7}} = c$

- Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Sol. Here $M = y(xy + 2x^2y^2)$ and $N = x(xy - x^2y^2)$

$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2$ and $\frac{\partial N}{\partial x} = 2xy - 3x^2y^2$

Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence equation is not exact.

Since the given equation is of the form $(xy)ydx + g(xy)xdy = 0$, I.F. = $\frac{1}{Mx-Ny} = \frac{1}{3x^3y^3}$

Multiplying the given equation by $\frac{1}{3x^3y^3}$, we get $\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$

Now $M = \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)$and $N = \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)$

$\frac{\partial M}{\partial y} = \frac{-1}{3x^2y^2}$ and $\frac{\partial N}{\partial x} = \frac{-1}{3x^2y^2}$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$\xrightarrow{yields} \int \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int \left(-\frac{1}{3y}\right)dy = c \xrightarrow{yields} \frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$

- Solve $(3xy^2 - y^3)dx - (2x^2y - xy^2)dy = 0$

Sol. Here $M = 3xy^2 - y^3$ and $N = -2x^2y + xy^2$

$\frac{\partial M}{\partial y} = 6xy - 3y^2$ and $\frac{\partial N}{\partial x} = -4xy + y^2$

Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence equation is not exact.

Since the given equation is homogenous, so

I.F. $= \frac{1}{Mx+Ny} = \frac{1}{3x^2y^2 - xy^3 - 2x^2y^2 + xy^3} = \frac{1}{x^2y^2}$

Multiplying the given equation by $\frac{1}{x^2y^2}$, we get

$\left(\frac{3}{x} - \frac{y}{x^2}\right)dx - \left(\frac{2}{y} - \frac{1}{x}\right)dy = 0$

Now $M = \left(\frac{3}{x} - \frac{y}{x^2}\right)$ and $N = \left(\frac{-2}{y} + \frac{1}{x}\right)$

$\frac{\partial M}{\partial y} = \frac{-1}{y^2}$and $\frac{\partial N}{\partial x} = \frac{-1}{y^2}$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$\xrightarrow{yields} \int \left(\frac{3}{x} - \frac{y}{x^2}\right)dx + \int \left(\frac{-2}{y}\right)dy = c \xrightarrow{yields} 3\log x + \frac{y}{x} + 3 - 2\log y = c$

- Solve $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$

Sol. Here $M = (x^2y^2 + \text{xy} + 1)y$ and $N = (x^2y^2 - \text{xy} + 1)x$

$\frac{\partial M}{\partial y} = 3x^2y^2 + 2\text{xy} + 1$ and $\frac{\partial N}{\partial x} = 3x^2y^2 - 2\text{xy} + 1$

Therefore $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ Hence equation is not exact.

Since the given equation is of the form $(xy)ydx + g(xy)xdy = 0$, I.F. = $\frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}$

Multiplying the given equation by $\frac{1}{2x^2y^2}$, we get $\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$

Now $M = \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)$and $N = \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)$

$\frac{\partial M}{\partial y} = \frac{-1}{3x^2y^2}$ and $\frac{\partial N}{\partial x} = \frac{-1}{3x^2y^2}$

Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence equation is exact.

Solution is given by $\int Mdx + \int terms\ of\ N\ free\ from\ x\ dy = c$

$$\xrightarrow{yields} \int \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int \left(-\frac{1}{3y}\right)dy = c \xrightarrow{yields} \frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

Equations solvable for y

- Solve $y + px = x^4p^2$

Sol. Given equation is $y + px = x^4p^2 \xrightarrow{yields} y = x^4p^2 - px$ ....(i)

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = 4x^3p^2 + 2x^4p\frac{dp}{dx} - p - x\frac{dp}{dx} \xrightarrow{yields} p = 4x^3p^2 + 2x^4p\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$\xrightarrow{yields} 4x^3p^2 + 2x^4p\frac{dp}{dx} - 2p - x\frac{dp}{dx} = 0 \xrightarrow{yields} 2x^3p\left(2p + x\frac{dp}{dx}\right) - \left(2p + x\frac{dp}{dx}\right) = 0$$

$$\xrightarrow{yields} (2x^3p - 1)\left(2p + x\frac{dp}{dx}\right) = 0 \xrightarrow{yields} \left(2p + x\frac{dp}{dx}\right) = 0$$

(By neglecting the factor$(2x^3p - 1) = 0$ *because it doesnot contain* $\frac{dp}{dx}$)

$$\xrightarrow{yields} 2\frac{dx}{x} + \frac{dp}{p} = 0 \xrightarrow{yields} p = \frac{c}{x^2}$$

Putting the value of p in (i), we get $y = x^4\frac{c^2}{x^4} - \frac{c}{x^2}x$

$\xrightarrow{yields}$ $y = c^2 - \frac{c}{x}$is the required solution

Equations solvable for x

- Solve $y = 2xp + y^2p^3$, where $p = \frac{dy}{dx}$

  Sol. $y = 2xp + y^2p^3 \xrightarrow{yields} x = \frac{1}{2}\left(\frac{y}{p} - y^2p^2\right)$

  Differentiating both sides w.r.t y, we get

$$\frac{1}{p} = \frac{1}{2}\left(\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - 2y^2p\frac{dp}{dy}\right) \xrightarrow{yields} 2p = p - y\frac{dp}{dy} - 2yp^4 - 2y^2p^3\frac{dp}{dy}$$

$$\xrightarrow{yields} p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0 \xrightarrow{yields} \left(p + y\frac{dp}{dy}\right)(1 + 2yp^3) = 0$$

$$\xrightarrow{yields} \left(p + y\frac{dp}{dy}\right) = 0$$

  (By neglecting the factor $(1 + 2yp^3) = 0$ *because it doesnot contain* $\frac{dp}{dy}$)

$$\xrightarrow{yields} \frac{dp}{p} + \frac{dy}{y} = 0 \xrightarrow{yields} logp + log\ y = logc \xrightarrow{yields} py = c \xrightarrow{yields} p = \frac{c}{y}$$

  By putting in given equation, we get $y^2 = 2cx + c^3$

Linear Differential Equations

- **Linear Differential equation:** A differential equation is said to be linear differential equation if it is of degree one and does not contain any term having product of dependent variable and its derivative.
- **Leibnitz form :** An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is called Leibnitz form.

  Integrating factor $= e^{\int P(x)dx}$ and

  Solution of given equation is $\text{(Integrating factor)z} = \int Q(x)\ \text{(Integrating factor)dx} + \text{c}$

- **Bernoulli's form:** An equation of the form $\frac{dy}{dx} + P(x)y = y^n Q(x)$ is called Bernoulli's form.

  Dividing both sides by y^n, we get $y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \ldots\ldots\ldots\ldots\ldots\ldots..(i)$

  Put $y^{1-n} = z \xrightarrow{yields} (1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx} \xrightarrow{yields} y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dz}{dx}$.

  $\therefore (i) becomes\ \frac{1}{(1-n)}\frac{dz}{dx} + P(x)z = Q(x) \xrightarrow{yields} \frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$

It is Leibnitz form.

- **Higher Order Equations**

  Let the differential equation be of the form $F(D)y = f(x)$

  Auxiliary eq. is given by $F(D) = 0$. Solve it.

Rules for Complementary Function:

1. If the roots of Auxiliary equation are real and distinct say m_1, m_2, m_3 , then
$$C.F. = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x}$$
2. If the roots of Auxiliary equation are real and distinct say m, m, m, m, m , then
$$C.F. = c_1e^{mx} + xc_2e^{mx} + x^2c_3e^{mx} + x^3c_4e^{mx} + x^4c_5e^{mx}$$
3. If the roots are a pair of complex conjugate numbers say $\alpha \pm i\beta$, then
$$C.F. = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$
4. If the roots are repeated pair of complex conjugate numbers say $\alpha \pm i\beta, \alpha \pm i\beta, \alpha \pm i\beta$ then
$$C.F. = e^{\alpha x}[(c_1 + xc_2 + x^2c_3)\cos\beta x + (c_1 + xc_2 + x^2c_3)\sin\beta x]$$

Rules for Particular integral: Let the equation be of the form $F(D)y = f(x)$.

$$P.I. = \frac{1}{F(D)}f(x)$$

1. If $f(x, y) = e^{ax}$
$$P.I. = \frac{1}{F(D)}e^{ax}$$
Put $D = a$ provided denominator does not become zero. If denominator becomes zero, then

$$P.I. = x \frac{1}{\frac{d}{dD}F(D)} e^{ax}$$

Put $D = a$ provided denominator does not become zero. If denominator becomes zero, then

$$P.I. = x^2 \frac{1}{\frac{d^2}{dD^2}F(D)} e^{ax}$$

Put $D = a$ provided denominator does not become zero.
Continue this process till we get a non-zero denominator.

2. If $f(x) = \cos(ax) \quad or \quad \sin(ax)$

$$P.I. = \frac{1}{F(D)} f(x)$$

Put $D^2 = -(a^2)$ provided denominator does not become zero. If denominator becomes zero, then

$$P.I. = x \frac{1}{\frac{d}{dD}F(D)} f(x)$$

Put $D^2 = -(a^2)$ provided denominator does not become zero. If denominator becomes zero, then

$$P.I. = x^2 \frac{1}{\frac{d^2}{dD^2}F(D)} f(x)$$

Put $D^2 = -(a^2)$ provided denominator does not become zero.

3. If $f(x) = x^m$

$$P.I. = \frac{1}{F(D)} x^m$$

Use Binomial Theorem i.e.

If $|x| < 1$, $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \ldots \ldots \ldots \ldots$

i. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots \ldots \ldots \ldots \ldots$
ii. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots \ldots \ldots \ldots \ldots$
iii. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \ldots \ldots \ldots \ldots$
iv. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \ldots \ldots \ldots \ldots$

4. If $f(x) = e^{ax}V$, where V is any function.

$$P.I. = \frac{1}{F(D)} e^{ax}V = e^{ax}\left(\frac{1}{F(D)}V\right)$$

5. General Rule: Factorize $F(D)$ into linear factors of the form $(D - A)$. Apply partial fractions.

$$\frac{1}{(D - A)} f(x) = e^{Ax} \int e^{-Ax} f(x) dx$$

Method of Variation of Parameters

It is a special method to find particular integral of a second order differential equation.

i. Calculate the complementary function. Let it be $c_1y_1 + c_2y_2$
ii. Calculate $W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix}$
iii. $u = -\int \frac{y_2 X dx}{W}$ and $v = \int \frac{y_1 X dx}{W}$
iv. Particular Integral = $uy_1 + vy_2$

Cauchy's homogenous equation:

An equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots \ldots \ldots .. + a_1 x^1 \frac{d^1 y}{dx^1} + a_0 y = f(x)$$

Is called Cauchy's homogenous equation.

Put $x = e^z \xrightarrow{yields} z = \log x \xrightarrow{yields} \frac{dz}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \xrightarrow{yields} \frac{dy}{dx} = \frac{dy}{dz}\frac{1}{x} \xrightarrow{yields} x\frac{dy}{dx} = \frac{dy}{dz} \xrightarrow{yields} x\frac{dy}{dx} = Dy \text{ where } D = \frac{d}{dz}$$

$$Similarly\ we\ get\ x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

Legendre's equation:

An equation of the form

$$a_n (ax+b)^n \frac{d^n y}{dx^n} + a_{n-1} (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots \ldots \ldots .. + a_1 (ax+b)^1 \frac{d^1 y}{dx^1} + a_0 y = f(x)$$

Is called Cauchy's homogenous equation.

Put $(ax+b) = e^z \xrightarrow{yields} z = \log(ax+b) \xrightarrow{yields} \frac{dz}{dx} = \frac{a}{(ax+b)}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \xrightarrow{yields} \frac{dy}{dx} = \frac{dy}{dz}\frac{a}{(ax+b)} \xrightarrow{yields} (ax+b)\frac{dy}{dx} = a\frac{dy}{dz} \xrightarrow{yields} (ax+b)\frac{dy}{dx} = aDy \text{ where } D = \frac{d}{dz}$$

$$Similarly\ we\ get\ (ax+b)^2 \frac{d^2 y}{dx^2} = a^2 D(D-1)y$$

- Solve $(x-a)\frac{dy}{dx} + 3y = 12\,(x-a)^3$

Sol. Given equation is

$$(x-a)\frac{dy}{dx}+3y=12\,(x-a)^3 \xrightarrow{yields} \frac{dy}{dx}+\frac{3y}{(x-a)}=12\,(x-a)^2$$

It is Leibnitz form.

Here $P(x)=\frac{3}{(x-a)}$ and $Q(x)=12\,(x-a)^2$

Integrating factor is $e^{\int P(x)dx}=e^{\int\frac{3}{(x-a)}dx}=e^{3ln(x-a)}=(x-a)^3$

∴ Solution is (Integrating factor)z = $\int Q(x)$ (Integrating factor)dx + c

$$\xrightarrow{yields}(x-a)^3y=\int(x-a)^3 12\,(x-a)^2\text{dx}+\text{c} \xrightarrow{yields}(x-a)^3y=\int 12\,(x-a)^5\text{dx}+\text{c}$$

$$\xrightarrow{yields}(x-a)^3y=\frac{12\,(x-a)^6}{6}+c \xrightarrow{yields}(x-a)^3y=2(x-a)^6+c \xrightarrow{yields} y=2(x-a)^3+\frac{c}{(x-a)^3}$$

- Write Bernoulli's equation.

  Sol. The equation $\frac{dy}{dx}+P(x)y=Q(x)y^n$ is called Bernoulli's equation.

- Solve $\frac{dy}{dx}+4xy+xy^3=0$

Sol. Given equation is

$$\frac{dy}{dx}+4xy+xy^3=0 \xrightarrow{yields}\frac{dy}{dx}+4xy=-xy^3 \xrightarrow{yields} y^{-3}\frac{dy}{dx}+4xy^{-2}=-x$$

Put $y^{-2}=z \xrightarrow{yields} -2y^{-3}\frac{dy}{dx}=\frac{dz}{dx}\xrightarrow{yields} y^{-3}\frac{dy}{dx}=\frac{-1}{2}\frac{dz}{dx}$

Equation becomes $\frac{-1}{2}\frac{dz}{dx}+4xz=-x \xrightarrow{yields}\frac{dz}{dx}-8xz=2x$. It is Leibnitz form.

Here $P(x)=-8x$ and $Q(x)=2x$

Integrating factor is $e^{\int P(x)dx}=e^{\int(-8x)dx}=e^{-4x^2}$

∴ Solution is (Integrating factor)z = $\int Q(x)$ (Integrating factor)dx + c

$$\xrightarrow{yields} ze^{-4x^2}=\int 2x\left(e^{-4x^2}\right)\text{dx}+\text{c} \xrightarrow{yields} y^{-2}e^{-4x^2}=\int\left(e^{-4t}\right)\text{dt}+\text{c}$$

$$\xrightarrow{yields} y^{-2}e^{-4x^2}=\frac{e^{-4t}}{-4}+c \xrightarrow{yields} y^{-2}e^{-4x^2}=\frac{e^{-4x^2}}{-4}+c \xrightarrow{yields} y^{-2}=\frac{-1}{4}+ce^{-4x^2}$$

- Solve $\frac{dy}{dx}-y=y^2(\sin x+\cos x)$

Sol. Given equation is

$$\frac{dy}{dx} - y = y^2(\sin x + \cos x) \xrightarrow{yields} y^{-2}\frac{dy}{dx} - y^{-1} = (\sin x + \cos x)$$

Put $y^{-1} = z \xrightarrow{yields} -y^{-2}\frac{dy}{dx} = \frac{dz}{dx} \xrightarrow{yields} y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$

Equation becomes $-\frac{dz}{dx} - z = (\sin x + \cos x) \xrightarrow{yields} \frac{dz}{dx} + z = -(\sin x + \cos x)$. It is Leibnitz form.

Here $P(x) = 1$ and $Q(x) = -(\sin x + \cos x)$

Integrating factor is $e^{\int P(x)dx} = e^{\int 1dx} = e^x$

∴ Solution is (Integrating factor)z $= \int Q(x)$ (Integrating factor)dx + c

$$\xrightarrow{yields} ze^x = \int -e^x(\sin x + \cos x)\text{dx} + \text{c} \xrightarrow{yields} \frac{e^x}{y} = -\left[\frac{e^x}{2}(\sin x - \cos x) + \frac{e^x}{2}(\cos x + \sin x)\right] + \text{c}$$

$$\xrightarrow{yields} \frac{1}{y} = -\sin x + \frac{c}{e^x}$$

- Solve $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

Sol. Given equation is $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}} \xrightarrow{yields} \frac{dx}{dy} = \frac{x+\sqrt{xy}}{y} \xrightarrow{yields} \frac{dx}{dy} - \frac{x}{y} = \frac{\sqrt{x}}{\sqrt{y}} \xrightarrow{yields} x^{-1/2}\frac{dx}{dy} - \frac{x^{1/2}}{y} = \frac{1}{\sqrt{y}}$

Put $x^{1/2} = z \xrightarrow{yields} x^{-1/2}\frac{dx}{dy} = 2\frac{dz}{dy}$

Equation becomes $2\frac{dz}{dy} - \frac{z}{y} = \frac{1}{\sqrt{y}} \xrightarrow{yields} \frac{dz}{dy} - \frac{z}{2y} = \frac{1}{2\sqrt{y}}$. It is Leibnitz form.

Here $P(y) = -\frac{1}{2y}$ and $Q(y) = \frac{1}{2\sqrt{y}}$

Integrating factor is $e^{\int P(y)dy} = e^{\int\left(-\frac{1}{2y}\right)dy} = e^{log(1/\sqrt{y})=}\frac{1}{\sqrt{y}}$

∴ Solution is (Integrating factor)z $= \int Q(y)$ (Integrating factor)dy + c

$$\xrightarrow{yields} \frac{z}{\sqrt{y}} = \int \frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{y}}\right)\text{dy} + \text{c} \xrightarrow{yields} \frac{\sqrt{x}}{\sqrt{y}} = \frac{1}{2}\text{logy} + \text{c} \xrightarrow{yields} \sqrt{x} = \frac{1}{2}\sqrt{y}\ \text{logy} + \text{c}\sqrt{y}$$

- Solvetan $y\frac{dy}{dx} + \tan x = \cos y\, cos^2 x$.

Sol. $\tan y\frac{dy}{dx} + \tan x = \cos y\, cos^2 x$

Dividing both sides by $\cos y$, we get $\sec y\ \tan y \frac{dy}{dx} + \sec y \tan x = \ cos^2 x$

Put $\sec y = z \xrightarrow{yields} \sec y\ \tan y \frac{dy}{dx} = \frac{dz}{dy}$

Equation becomes $\frac{dz}{dy} - z \tan x = \ cos^2 x$. It is Leibnitz form.

Here $P(x) = -\tan x$ and $Q(x) = cos^2 x$

Integrating factor is $e^{\int P(x)dx} = e^{\int -\tan x dx} = e^{log(\cos x)} = \cos x$

$\therefore$ Solution is (Integrating factor)z $= \int Q(x)$ (Integrating factor)dx + c

$$\xrightarrow{yields} z \cos x = \int \cos x\, cos^2 x\ \mathrm{dx} + \mathrm{c} \xrightarrow{yields} z \cos x = \int cos^3 x\ \mathrm{dx} + \mathrm{c}$$

$$\xrightarrow{yields} z \cos x = \frac{1}{4}\int (\cos 3\mathrm{x} + 3\cos \mathrm{x})\ \mathrm{dx} + \mathrm{c} \xrightarrow{yields} z \cos x = \frac{1}{4}\left[\frac{\sin 3\mathrm{x}}{3} + 3\sin \mathrm{x}\right] + \mathrm{c}$$

$$\xrightarrow{yields} \sec y \cos x = \frac{1}{4}\left[\frac{\sin 3\mathrm{x}}{3} + 3\sin \mathrm{x}\right] + \mathrm{c}$$

- Solve$y'' - 4y' - 5y = 0$

Sol. The given equation is $y'' - 4y' - 5y = 0$

Auxiliary equation is $D^2 - 4D - 5 = 0 \xrightarrow{yields} D^2 - 5D + D - 5 = 0 \xrightarrow{yields} (D-5)(D+1) = 0$

$$\xrightarrow{yields} D = -1, 5$$

Therefore complementary function is $c_1 e^{-x} + c_2 e^{5x}$

Particular Integral = 0

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1 e^{-x} + c_2 e^{5x}$

- Solve $y'' + 2y' + 2y = 0$

Sol. The given equation is $y'' + 2y' + 2y = 0$

Auxiliary equation is $D^2 + 2D + 2 = 0$

$$\xrightarrow{yields} D = -1 \pm i$$

Therefore complementary function is $e^{-x}(c_1 \cos x + c_2 sinx)$

Particular Integral = 0

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = e^{-x}(c_1 \cos x + c_2 sinx)$

- Solve $y'''' + 32y'' + 256y = 0$
  Sol. The given equation is $y'''' + 32y'' + 256y = 0$

  Auxiliary equation is $D^4 + 32D^2 + 256 = 0 \xrightarrow{yields} (D + 16)^2 = 0$

  $$\xrightarrow{yields} D = \pm 4i, \pm 4i, \pm 4i, \pm 4i$$

  Therefore complementary function is

  $$[(c_1 + c_2x + c_3x^2 + c_4x^4) \cos 4x + (c_5 + c_6x + c_7x^2 + c_8x^4) sin4x]$$

  Particular Integral = 0

  Complete Solution = Complementary function + Particular Integral

  $$\xrightarrow{yields} y = [(c_1 + c_2x + c_3x^2 + c_4x^4) \cos 4x + (c_5 + c_6x + c_7x^2 + c_8x^4) sin4x]$$

- Solve the differential equation $y''' - 3y' - 2y = 0$

Sol. The given equation is $y''' - 3y' - 2y = 0$

Auxiliary equation is $D^3 - 3D - 2 = 0 \xrightarrow{yields} (D + 1)^2(D - 2) = 0$

$$\xrightarrow{yields} D = -1, -1, 2$$

Therefore complementary function is $c_1e^{-x} + c_2xe^{-x} + c_3e^{2x}$

Particular Integral = 0

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1e^{-x} + c_2xe^{-x} + c_3e^{2x}$

- Solve $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, given x(0)=0 and x'(0)=15

Sol. Auxiliary equation is $D^2 + 5D + 6 = 0 \xrightarrow{yields} D = -2, -3$

Therefore complementary function is $c_1e^{-2t} + c_2e^{-3t}$

Particular Integral= $\frac{1}{D^2+5D+6}(0) = 0$

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} x = c_1 e^{-2t} + c_2 e^{-3t}$

Therefore $x' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$

Given x(0)=0 $\xrightarrow{yields} c_1 + c_2 = 0 \xrightarrow{yields} c_1 = -c_2$

and x'(0)=15 $\xrightarrow{yields}$ $2c_1 + 3c_2 = -15 \xrightarrow{yields} -2c_2 + 3c_2 = -15 \xrightarrow{yields} c_2 = -15 \xrightarrow{yields} c_1 = 15$

Hence complete solution is $x = 15e^{-2t} + -15e^{-3t}$

- Find a differential equation of the form $ay'' + by' + cy = 0$, for which e^{-x} and xe^{-x} are solutions.
  Sol. The required equation is $(D+1)^2 y = 0$
- Solve$(4D^2 - 4D + 1)y = e^{x/2}$

Sol. The given equation is $(4D^2 - 4D + 1)y = e^{x/2}$

Auxiliary equation is $4D^2 - 4D + 1 = 0 \xrightarrow{yields} (2D-1)^2 y = 0 \xrightarrow{yields} D = \frac{1}{2}, \frac{1}{2}$

Therefore complementary function is $= \left(c_1 e^{x/2} + xc_2 e^{x/2}\right)$

Particular Integral $= \frac{1}{(2D-1)^2} e^{x/2} \left(Put\ D = \frac{1}{2}\right)$

But it is case of failure.

$\therefore$ Particular Integral $= x\frac{1}{2(2D-1)2} e^{x/2} \left(Put\ D = \frac{1}{2}\right)$

But it is again case of failure.

$\therefore$ Particular Integral $= x^2 \frac{1}{8} e^{x/2} = \frac{x^2 e^{x/2}}{8}$

$\therefore$ Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields}$ $y = \left(c_1 e^{x/2} + xc_2 e^{x/2}\right) + \frac{x^2 e^{x/2}}{8}$

- Solve $(D^2\text{-}3D\text{+}2)y\text{=}2e^x\cos\frac{x}{2}$

Sol. Auxiliary equation is $D^2 - 3D + 2 = 0 \xrightarrow{yields} D = 2, 1$
Therefore complementary function is $c_1 e^{2x} + c_2 e^x$

Particular Integral $= \frac{1}{D^2-3D+2} 2e^x \cos\frac{x}{2} = 2e^x \left[\frac{1}{((D+1)^2-3(D+1)+2)} \cos\frac{x}{2}\right] = 2e^x \left[\frac{1}{(D^2-D)}\right] \cos\frac{x}{2}$

$\left(Put\ D^2 = -\frac{1}{4}\right) =$ $2e^x \left[\frac{1}{\left(-\frac{1}{4}-D\right)}\right] \cos\frac{x}{2} = -8e^x \left[\frac{1}{1+4D}\right] \cos\frac{x}{2}$

$$= -8e^x\left[\frac{1-4D}{1-16D^2}\right]\cos\frac{x}{2}\left(Put\ D^2 = -\frac{1}{4}\right)$$

$= -\frac{8}{5}e^x\left(\cos\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2}\right)$

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1e^{2x} + c_2e^x - \frac{8}{5}e^x\left(\cos\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2}\right)$

- Solve $y'' - 2y' + y = x\sin x$

Sol. The given equation is $y'' - 2y' + y = x\sin x$

Auxiliary equation is $D^2 - 2D + 1 = 0 \xrightarrow{yields} D = 1,1$

Therefore complementary function is $c_1e^x + xc_2e^x$

Particular Integral $= \frac{1}{(D-1)^2}x\sin x = x\left[\frac{1}{D^2-2D+1}\sin x\right] - \left[\frac{2(D-1)}{D^2-2D+1}\sin x\right] (Put\ D^2 = -1)$

$$= x\left[\frac{1}{-2D}\sin x\right] - \left[\frac{2(D-1)}{-2D}\sin x\right] = \frac{1}{2}x\cos x - (D-1)\cos x = \frac{1}{2}x\cos x + \sin x + \cos x$$

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1e^x + xc_2e^x + \frac{1}{2}x\cos x + \sin x + \cos x$

- Solve$(D^2 + 3D + 2)y = xe^x\sin x$

Sol. Auxiliary equation is $D^2 + 3D + 2 = 0 \xrightarrow{yields} D = -1, -2$

Therefore complementary function is $c_1e^{-x} + c_2e^{-2x}$

Particular Integral $= \frac{1}{D^2+3D+2}xe^x\sin x = e^x\left[\frac{1}{((D+1)^2+3(D+1)+2)}x\sin x\right] = e^x\left[\frac{1}{(D^2+5D+6)}x\sin x\right]$

$= e^x\left[x\left[\frac{1}{D^2+5D+6}\sin x\right] - \left[\frac{2D+5}{(D^2+5D+6)^2}\sin x\right]\right] (Put\ D^2 = -1)$

$$= e^x\left[x\left[\frac{1}{5D+5}\sin x\right] - \left[\frac{2D+5}{(5D+5)^2}\sin x\right]\right]$$

$$= e^x\left[\frac{x}{5}\left[\frac{(D-1)}{D^2-1}\sin x\right] - \frac{1}{25}\left[\frac{(2D+5)}{D^2+2D+1}\sin x\right]\right] (Put\ D^2 = -1)$$

$= e^x\left[\frac{-x}{10}[(D-1)\sin x] - \frac{1}{25}\left[\frac{(2D+5)}{2D}\sin x\right]\right] = e^x\left[\frac{-x}{10}[(D-1)\sin x] + \frac{1}{50}[(2D+5)\cos x]\right]$

$$= e^{x}\left[\frac{-x}{10}(\cos x - \sin x) + \frac{1}{50}[(-2\sin x + 5\cos x)]\right]$$

Complete Solution = Complementary function + Particular Integral

$$\xrightarrow{yields} y = c_1e^{-x} + c_2e^{-2x} + e^{x}\left[\frac{-x}{10}(\cos x - \sin x) + \frac{1}{50}[(-2\sin x + 5\cos x)]\right]$$

- Using differential operator, find general solution of $(D^2 + 9)y = xe^{2x}\cos x$

Sol. Auxiliary equation is $D^2 + 9 = 0 \xrightarrow{yields} D = \pm 3i$

Therefore complementary function is $(c_1\cos 3x + c_2\sin 3x)$

Particular Integral $= \frac{1}{D^2+9}xe^{2x}\cos x = e^{2x}\left[\frac{1}{((D+2)^2+9)}x\cos x\right] = e^{2x}\left[\frac{1}{(D^2+4D+13)}x\cos x\right]$

$$= e^{2x}\left[x\left[\frac{1}{D^2+4D+13}\cos x\right] - \left[\frac{(2D+4)}{(D^2+4D+13)^2}\cos x\right]\right] \qquad (Put\ D^2 = -1)$$

$$= \left[e^{2x}x\left[\frac{1}{4D+12}\cos x\right] - \left[\frac{(2D+4)}{(4D+12)^2}\cos x\right]\right]$$

$$= e^{2x}\left[\frac{x}{4}\left[\frac{(D-3)}{D^2-9}\cos x\right] - \frac{1}{16}\left[\frac{(2D+4)}{D^2+6D+9}\cos x\right]\right] (Put\ D^2 = -1)$$

$$= e^{2x}\left[\frac{x}{4}\left[\frac{(D-3)}{-10}\cos x\right] - \frac{1}{16}\left[\frac{(2D+4)}{6D+8}\cos x\right]\right]$$

$$= e^{2x}\left[\frac{x}{4}\left[\frac{(-\sin x - 3\cos x)}{-10}\right] - \frac{1}{16}\left[\frac{(2D+4)(6D-8)}{(36D^2-64)}\cos x\right]\right] \ (Put\ D^2 = -1)$$

$$= e^{2x}\left[\frac{x}{4}\left[\frac{(-\sin x - 3\cos x)}{-10}\right] + \frac{1}{1600}[(12D^2 + 8D - 32)\cos x]\right]$$

$$= e^{2x}\left[\frac{x}{40}(\sin x + 3\cos x) + \frac{1}{1600}(-12\cos x - 8\sin x - 32\cos x)\right]$$

$$= e^{2x}\left[\frac{x}{40}(\sin x + 3\cos x) - \frac{1}{1600}(44\cos x + 8\sin x)\right]$$

Complete Solution = Complementary function + Particular Integral

$$\xrightarrow{yields} y = (c_1\cos 3x + c_2\sin 3x) + e^{2x}\left[\frac{x}{40}(\sin x + 3\cos x) - \frac{1}{1600}(44\cos x + 8\sin x)\right]$$

METHOD OF VARIATION OF PARAMETERS

- Find Wroskian of 1 , $\sin x$ *and* $\cos x$.

Sol. Here Here $y_1 = 1$, $y_2 = \sin x$ *and* $y_3 = \cos x$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = -1$$

- Solve$y'' + 16y = 32 \sec 2x$

Sol. The given equation is $y'' + 16y = 32 \sec 2x$by method of variation of parameters.

Auxiliary equation is $D^2 + 16 = 0 \xrightarrow{yields} D = \pm 4i$

Therefore complementary function is $(c_1 \cos 4x + c_2 \sin 4x)$

Here $y_1 = \cos 4x$ *and* $y_2 = \sin 4x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4$$

$$u = -\int \frac{y_2 X dx}{W} = -\int \frac{\sin 4x 32 \sec 2x\, dx}{4} = -8\int 2\sin 2x cos 2x \sec 2x\, dx$$

$$= -16\int \sin 2x\, dx = 8\cos 2x$$

$$v = \int \frac{y_1 X dx}{W} = \int \frac{\cos 4x 32 \sec 2x dx}{4} = 8\int (2cos^2 2x - 1)\sec 2x\, dx$$

$$= 8\int (2\cos 2x - \sec 2x)\, dx = 8\sin 2x - 4log|\sec 2x + \tan 2x|$$

Particular Integral = $uy_1 + vy_2 = 8\cos 2x \cos 4x + (8\sin 2x - 4log|\sec 2x + \tan 2x|)\sin 4x$

$$= 8\cos 2x - 4\sin 4x\, log|\sec 2x + \tan 2x|$$

Complete Solution = Complementary function + Particular Integral

$$\xrightarrow{yields} y = (c_1 \cos 4x + c_2 \sin 4x) + 8\cos 2x - 4\sin 4x\, log|\sec 2x + \tan 2x|$$

- Solve$y'' + y = cosec\ x$ by method of variation of parameters.

Sol.The given equation is $y'' + y = cosec\ x \xrightarrow{yields} (D^2 + 1)y = cosec\ x$

Auxiliary equation is $D^2 + 1 = 0 \xrightarrow{yields} D = \pm i$

Therefore complementary function is $c_1 \cos x + c_2 \sin x$

Here $y_1 = \cos x \ and \ y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$u = -\int \frac{y_2 X dx}{W} = -\int \frac{\sin x cosec\ x\ dx}{1} = -\int dx = -x$

$$v = \int \frac{y_1 X dx}{W} = \int \frac{\cos x\ cosec\ x\ dx}{1} = \int \cot x\, dx = \log \sin x$$

Particular Integral = $uy_1 + vy_2 = -x\cos x + \sin x \log \sin x$

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1 \cos x + c_2 \sin x + (-x\cos x + \sin x \log \sin x)$

- Solve $y'' - 2y' + 2y = e^x \tan x$by method of variation of parameters.

Sol. The given equation is $y'' - 2y' + 2y = e^x \tan x$

Auxiliary equation is $D^2 - 2D + 2 = 0 \xrightarrow{yields} D = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

Therefore complementary function is $e^x(c_1 \cos x + c_2 \sin x)$

Here $y_1 = e^x \cos x \ and \ y_2 = e^x \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix} = e^{2x}$$

$u = -\int \frac{y_2 X dx}{W} = -\int \frac{e^x \sin x e^x \tan x\ dx}{e^{2x}} = -\int(\sec x - \cos x)dx = -(log|\sec x + \tan x| - \sin x)$

$$v = \int \frac{y_1 X dx}{W} = \int \frac{e^x \cos x\ \ e^x \tan x\, dx}{e^{2x}} = \int \sin x\, dx = -\cos x$$

Particular Integral = $uy_1 + vy_2 = -(log|\sec x + \tan x| - \sin x)e^x \cos x - \cos x\, e^x \sin x$

$= -e^x \cos x\, log|\sec x + \tan x|$

- Solve $y'' - y = \frac{2}{1+e^x}$by method of variation of parameters.

Sol. $y'' - y = \frac{2}{1+e^x}$

Auxiliary equation is $D^2 - 1 = 0 \xrightarrow{yields} D = \pm 1$

Therefore complementary function is $c_1 e^{-x} + c_2 e^x$

Here $y_1 = e^{-x}\ and\ y_2 = e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = 2$$

$$u = -\int \frac{y_2 X dx}{W} = -\int \frac{e^x}{2} \frac{2}{(1+e^x)} dx = -\int \frac{e^x}{(1+e^x)} dx = -\log(1+e^x)$$

$v = \int \frac{y_1 X dx}{W} = \int \frac{e^{-x}}{2} \frac{2}{(1+e^x)} dx = \int \frac{e^{-x}}{(1+e^x)} dx$ Put $e^{-x} = t$. $\therefore e^{-x} dx = -dt$

$$\int \frac{e^{-x}}{(1+e^x)} dx = -\int \frac{t^2}{(t+1)} dt = -\int \frac{t^2-1+1}{(t+1)} dt = -\int (t-1+\frac{1}{(t+1)}) dt = -\frac{t^2}{2} + t - \log((t+1)$$

$$= -\frac{e^{-2x}}{2} + e^{-x} - \log((e^{-x}+1)$$

Particular Integral = $uy_1 + vy_2 = -\log(1+e^x)\, e^{-x} + e^x \left[-\frac{e^{-2x}}{2} + e^{-x} - \log((e^{-x}+1)\right]$

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1 e^{-x} + c_2 e^x - \log(1+e^x)\, e^{-x} + e^x \left[-\frac{e^{-2x}}{2} + e^{-x} - \log((e^{-x}+1)\right]$

CAUCHY'S and LEGENDER'S EQUATION

- Solve$4x^2 y'' + y = 0$

Sol.$4x^2 y'' + y = 0$. It is Cauchy's homogenous equation. Put $x = e^z$

$$\therefore We\ get\ x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

Hence the given equation becomes $4D(D-1)y + y = 0 \xrightarrow{yields} (4D^2 - 4D + 1)y = 0$

Auxiliary equation is $4D^2 - 4D + 1 = 0 \xrightarrow{yields} D = \frac{1}{2}, \frac{1}{2}$

Therefore complementary function is $c_1 e^{\frac{1}{2}z} + z c_2 e^{\frac{1}{2}z} = \sqrt{x}(c_1 + c_2 \log x)$

Hence solution is $y = \sqrt{x}(c_1 + c_2 \log x)$

- Solve $x^2 y'' + 2xy' - 2y = 0$

Sol. The given equation is of the form $x^2 y'' + 2xy' - 2y = 0$

It is Cauchy's homogenous equation. Put $x = e^z \xrightarrow{yields} z = \log x \xrightarrow{yields} \frac{dz}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \xrightarrow{yields} \frac{dy}{dx} = \frac{dy}{dz}\frac{1}{x} \xrightarrow{yields} x\frac{dy}{dx} = \frac{dy}{dz} \xrightarrow{yields} x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$$Similarly\ we\ get\ x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Hence the given equation becomes $D(D-1)y + 2Dy - 2y = 0 \xrightarrow{yields} (D^2 + D - 2)y = 0$

Auxiliary equation is $D^2 + D - 2 = 0 \xrightarrow{yields} D = 1, -2$

Therefore complementary function is $c_1e^z + c_2e^{-2z} = c_1x + \frac{c_2}{x^2}$

Particular Integral $= 0$

∴ Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1x + \frac{c_2}{x^2}$

- Solve $x^2y'' + xy' + y = x$

Sol. The given equation is of the form $x^2y'' + xy' + y = x$

It is Cauchy's homogenous equation. Put $x = e^z \xrightarrow{yields} z = \log x \xrightarrow{yields} \frac{dz}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \xrightarrow{yields} \frac{dy}{dx} = \frac{dy}{dz}\frac{1}{x} \xrightarrow{yields} x\frac{dy}{dx} = \frac{dy}{dz} \xrightarrow{yields} x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$$Similarly\ we\ get\ x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Hence the given equation becomes $D(D-1)y + Dy + y = e^z \xrightarrow{yields} (D^2 + 1)y = e^z$

Auxiliary equation is $D^2 + 1 = 0 \xrightarrow{yields} D = \pm i$

Therefore complementary function is $c_1\cos z + c_2\sin z = c_1\cos(\log x) + c_2\sin(\log x)$

Particular Integral $= \frac{1}{D^2+1}e^z(Put\ D = 1)$

$= \frac{1}{2}e^z = \frac{x}{2}$

∴ Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1\cos(\log x) + c_2\sin(\log x) + \frac{x}{2}$

- Solve $x^2y'' + 5xy' + 3 = \ln x$

Sol. The given equation is of the form $x^2y'' + 5xy' + 3 = \ln x$

It is Cauchy's homogenous equation. Put $x = e^z \xrightarrow{yields} z = \ln x \xrightarrow{yields} \frac{dz}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \xrightarrow{yields} \frac{dy}{dx} = \frac{dy}{dz}\frac{1}{x} \xrightarrow{yields} x\frac{dy}{dx} = \frac{dy}{dz} \xrightarrow{yields} x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$$Similarly\ we\ get\ x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Hence the given equation becomes

$$D(D-1)y + 5Dy + 3y = z \xrightarrow{yields} (D^2 + 4D + 3)y = z$$

Auxiliary equation is $D^2 + 4D + 3 = 0 \xrightarrow{yields} D = -3, -1$

Therefore complementary function is $c_1e^{-z} + c_2e^{-3z} = \frac{c_1}{x} + \frac{c_2}{x^3}$

Particular Integral $= \frac{1}{D^2+4D+3}z = \frac{1}{3\left(1+\frac{D^2}{3}+\frac{4D}{3}\right)}z = \frac{1}{3}\left[1 + \left(\frac{D^2}{3} + \frac{4D}{3}\right)\right]^{-1} z$

$$= \frac{1}{3}\left[1 - \frac{D^2}{3} - \frac{4D}{3} + \left(\frac{D^2}{3} + \frac{4D}{3}\right)^2 - \cdots \dots \dots\right] z = \frac{1}{3}\left[z - 0 - \frac{4}{3} + 0 - 0 \dots \dots \dots\right]$$

$$= \frac{z}{3} - \frac{4}{9} = \frac{\ln x}{3} - \frac{4}{9}$$

∴ Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = \frac{c_1}{x} + \frac{c_2}{x^3} + \frac{\ln x}{3} - \frac{4}{9}$

- Solve Cauchy's homogenous linear eq. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = xlogx$

Sol. The given equation is of the form $x^2y'' - xy' + 2y = xlogx$

It is Cauchy's homogenous equation. Put $x = e^z \xrightarrow{yields} z = \log x \xrightarrow{yields} \frac{dz}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \xrightarrow{yields} \frac{dy}{dx} = \frac{dy}{dz}\frac{1}{x} \xrightarrow{yields} x\frac{dy}{dx} = \frac{dy}{dz} \xrightarrow{yields} x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$$Similarly\ we\ get\ x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Hence the given equation becomes

$$D(D-1)y - Dy + 2y = e^z z \xrightarrow{yields} (D^2 - 2D + 1)y = e^z z$$

Auxiliary equation is $D^2 - 2D + 1 = 0 \xrightarrow{yields} D = 1,1$

Therefore complementary function is $c_1 e^z + z c_2 e^z = c_1 x + c_2 x log x$

Particular Integral $= \frac{1}{D^2-2D+1} z e^z = e^z \left[\frac{1}{((D+1)^2-2(D+1)+1)} z\right]$

$$= e^z \left[\frac{1}{D^2} z\right] = e^z \frac{z^3}{6} = x \frac{(\log x)^3}{6}$$

∴ Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1 x + c_2 x log x + x \frac{(\log x)^3}{6}$

- Solve Cauchy's homogenous linear eq.$x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} - 5y = 24 x log x$

Sol. The given equation is of the form $x^2 y'' + 5xy' - 5y = x log x$

It is Cauchy's homogenous equation. Put $x = e^z \xrightarrow{yields} z = \log x \xrightarrow{yields} \frac{dz}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \xrightarrow{yields} \frac{dy}{dx} = \frac{dy}{dz}\frac{1}{x} \xrightarrow{yields} x\frac{dy}{dx} = \frac{dy}{dz} \xrightarrow{yields} x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$$Similarly\ we\ get\ x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

Hence the given equation becomes

$$D(D-1)y + 5Dy - 5y = 24 e^z z \xrightarrow{yields} (D^2 + 4D - 5)y = 24 e^z z$$

Auxiliary equation is $D^2 + 4D - 5 = 0 \xrightarrow{yields} D = -5, 1$

Therefore complementary function is $c_1 e^{-5z} + c_2 e^z = \frac{c_1}{x^5} + c_2 x$

Particular Integral $= \frac{1}{D^2+4D-5} 24 z e^z = 24 e^z \left[\frac{1}{((D+1)^2+4(D+1)-5)} z\right]$

$$= 24 e^z \left[\frac{1}{D^2 + 6D} z\right] = \frac{24 e^z}{6D}\left(1 + \frac{D}{6}\right)^{-1} z = \frac{4e^z}{D}\left(1 - \frac{D}{6} + \frac{D^2}{36} \ldots \ldots \ldots\right) z = \frac{4e^z}{D}\left(z - \frac{1}{6}\right)$$
$$= 4e^z\left(\frac{z^2}{2} - \frac{z}{6}\right) = 4x\left(\frac{(\log x)^2}{2} - \frac{\log x}{6}\right)$$

∴ Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = \frac{c_1}{x^5} + c_2 x + 4x\left(\frac{(\log x)^2}{2} - \frac{\log x}{6}\right)$

- Solve$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos[\log(1+x)]$

Sol. Put $(1+x)=e^z \xrightarrow{yields} z=\log(1+x)$

The given equation becomes $[D(D-1)+D+1]y=4\cos z \xrightarrow{yields} (D^2+1)y=4\cos z$

Auxiliary equation is $D^2+1=0 \xrightarrow{yields} D=\pm i$

Therefore complementary function is $=(c_1\cos z+c_2\sin z)$

$$=(c_1\cos\log(1+x)+c_2\sin\log(1+x))$$

Particular Integral $=\frac{1}{D^2+1}4\cos z\,(Put\ D^2=-1)$

But it is case of failure. $\therefore$ Particular Integral $=z\frac{1}{2D}4\cos z$

$=2zsinz=2\log(1+x)\sin\log(1+x)$

$\therefore$ Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y=(c_1\cos\log(1+x)+c_2\sin\log(1+x))+2\log(1+x)\sin\log(1+x)$

SIMULTANEOUS EQUATIONS

- Solve$(D+5)x+y=e^t$ and $x+(D+5)y=e^{5t}$ simultaneously.

Sol. Given equations are $(D+5)x+y=e^t$ …………..(i) and $x+(D+5)y=e^{5t}$ …………..(ii)

Multiplying (i) by (D+5) and subtracting (ii) from (i), we get

$$(D^2+10D+24)x=6e^t-e^{5t}$$

Auxiliary equation is $D^2+10D+24=0 \xrightarrow{yields} D=-4,-6$

Therefore complementary function is $c_1e^{-4t}+c_2e^{-6t}$

Particular Integral $=\frac{1}{D^2+10D+24}(6e^t-e^{5t})=\frac{6e^t}{35}-\frac{e^{5t}}{99}$

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} x=c_1e^{-4t}+c_2e^{-6t}+\frac{6e^t}{35}-\frac{e^{5t}}{99}$

Putting the value of x in (i), we get $(D+5)(c_1e^{-4t}+c_2e^{-6t}+\frac{6e^t}{35}-\frac{e^{5t}}{99})+y=e^t$

$\xrightarrow{yields} -4c_1e^{-4t}-6c_2e^{-6t}+\frac{6e^t}{35}-\frac{5e^{5t}}{99}+5c_1e^{-4t}+5c_2e^{-6t}+\frac{30e^t}{35}-\frac{5e^{5t}}{99}+y=e^t$

$$\xrightarrow{yields} c_1e^{-4t} - c_2e^{-6t} + \frac{36e^t}{35} - \frac{10e^{5t}}{99} + y = e^t \xrightarrow{yields} y = -c_1e^{-4t} + c_2e^{-6t} - \frac{e^t}{35} + \frac{10e^{5t}}{99}$$

- Solve D$x + 2y = -\sin t$ and $-2x + Dy = \cos t$ simultaneously.

Sol. Given equations are D$x + 2y = -\sin t$ …………..(i) and $-2x + Dy = \cos t$ …………..(ii)

Multiplying (i) by 2 and (ii) by D and adding (ii) and (i), we get

$$(D^2 - 4)y = -3\sin t$$

Auxiliary equation is $D^2 - 4 = 0 \xrightarrow{yields} D = -2, 2$

Therefore complementary function is $c_1e^{-2t} + c_2e^{2t}$

Particular Integral $= \frac{1}{(D^2-4)}(-3\sin t) = \frac{3\sin t}{5}$ (by putting $D^2 = -1$)

Complete Solution = Complementary function + Particular Integral

$\xrightarrow{yields} y = c_1e^{-2t} + c_2e^{2t} + \frac{3\sin t}{5}$

Putting the value of y in (ii), we get $-2x + D\left(c_1e^{-2t} + c_2e^{2t} + \frac{3\sin t}{5}\right) = \cos t$

$$\xrightarrow{yields} -2x + \left(-2c_1e^{-2t} + 2c_2e^{2t} + \frac{3\cos t}{5}\right) = \cos t$$

$$\xrightarrow{yields} x = \frac{\cos t}{5} + c_1e^{-2t} - c_2e^{2t}$$

SPECIAL FUNCTIONS

Q.1 Define error function and write its two properties.

Ans. Error function = $erf x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2 dt}$.

Properties:

1. $erf(-x) = -erf x$

2. $erf(x)+erf_c(x)=1$

Q.2 What are ordinary, regular & irregular singular points of an ordinary differential function?

Ans. Ordinary Point: A point $x=x_0$ is called an ordinary point of the equation:

$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$; if both functions $P(x)$ and $Q(x)$ are analytic at $x=x_0$.

Regular Singular Point: If both $(x-x_0)P(x)$ and $(x-x_0)^2 q(x)$ are analytic at $x=x_0$.

Irregular Singular Point: A singular point which is not regular, is called Irregular Singular Point.

Q.3 Explain the Power series solution to solve Differential equations.

Ans. Step – I Assume the solution in the form

$$y=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots\ldots\ldots$$

Step – IIFind $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Step – III Substitute the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given Differential equation.

Step – IV Equate to zero the coefficients of various power of x and find $a_2,a_3,a_4,a_5,\ldots\ldots\ldots\ldots$ in terms of a_0 and a_1.

Step – V Substitute the value of $a_2,a_3,a_4,a_5,\ldots\ldots\ldots\ldots$ in the assumed solution of y.

Q.4 What is Legendre's differential equation?

Ans. $$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

Q.5 What is Bessel's differential equation?

Ans. The differential equation of the form

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+n(n+1)y=0$$

Q.6 Explain the method of forbenius to solve the differential equation.

Ans. Step – I Assume $y=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\ldots\ldots\ldots\ldots$

Step – II Find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Step – III Substitute the value of $y,\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation.

Step – IV Equate the coefficient of lowest power of x equal to zero. This gives a quadratic equation in m, which is known as Indicial equation.

Step – V Equate the coefficient of powers of x to find $a_1,a_2,a_3,\ldots\ldots\ldots\ldots$ in terms of a_0.

Q.7 Write down the Rodrigue's formula.

Ans. $$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

Q.8 Show that $x^4 = \frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right]$.

Ans. $P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]$

$P_2(x) = \frac{1}{2}\left[3x^2 - 1\right]$

$P_0(x) = 1$

Hence $\frac{1}{35}\left[8\frac{1}{8}\left(35x^4 - 30x^2 + 3\right) + 20\frac{1}{2}\left(3x^2 - 1\right) + 7\right] = x^4$

Q.9 Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of Legendre's polynomials.

Ans. $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$; $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$; $x = P_1(x)$; $P_0(x) = 1$

$$x^3 - 5x^2 + x + 2 = \left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - 5\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + P_1(x) + 2P_0(x)$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x)$$

Q.10 Prove that $J_0'(x) = -J_1(x)$

Ans. $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$

Put n = 0, $\frac{d}{dx}[J_0(x)] = -J_1(x)$

$J_0'(x) = -J_1(x)$

Q.11 State the orthogonality condition of Bessel's function.

Ans. If α and $ß$ are the roots of $J_n(x) = 0$

$$\int_0^1 x J_n(\alpha x) J_n(ßx) dx = \begin{cases} 0 & \text{when } \alpha \neq ß \\ \frac{1}{2} J_{n+1}(\alpha) & \text{when } \alpha = ß \end{cases}$$

Q.12 Prove that $erf\,0 = 0$.

Ans. As we know $erf x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2 dt}$

Put $x = 0$

$$erf\,0 = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-t^2 dt} = 0$$

Q.13 What does $J_n(x)$ represent? What is the value of $J_{\frac{1}{2}}(x)$?

Ans. $J_n(x)$ represents the Bessel function which is the solution of the Bessel differential equation.

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

Q.14 Prove that $P_n(1) = 1$.

Ans. As we know, $\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-\frac{1}{2}}$

Put $x = 1$,

$$\sum_{n=0}^{\infty} h^n P_n(1) = (1 - 2h + h^2)^{-\frac{1}{2}}$$

$$= (1 - h)^{-1}$$

$$= 1 + h + h^2 + \ldots\ldots\ldots\ldots + h^n + \ldots\ldots\ldots\ldots$$

equate the coefficients of h^n, then $P(1) = 1$.

Q.15 Prove that $P_n(-x) = (-1)^n P_n(x)$.

Ans. As we know $\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-\frac{1}{2}}$ .... (1)

Replace x by $-x$ in (1),

$$\sum_{n=0}^{\infty} h^n P_n(-x) = (1 + 2xh + h^2)^{-\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (2)$$

Replace h by $-h$ in (1),

$$\sum_{n=0}^{\infty} (-h)^n P_n(x) = (1 + 2xh + h^2)^{-\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (3)$$

From (2) and (3),

$$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-h)^n P_n(x)$$

$$P_n(-x) = (-1)^n P_n(x)$$

Q.16 Stat the orthogonal property of Legendre's polynomials.

Ans. $$\int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$$

Q.17 Solve the differential equations in terms of Bessel function.

Ans. The given equation is $x^2 y'' + 2xy' + (4x^4 - 4)y = 0$

Comparing with general form, we get,

$$1 - 2\alpha = -2, \quad ß^2 r^2 = 4, \quad 2r = 4, \quad \alpha^2 - n^2 r^2 = -4$$

$$\alpha = \frac{3}{2}, \quad ß = 1, \; r = 2, \; n = \frac{5}{4}$$

n is not an integer. $y = x^{\frac{3}{2}} \left[c_1 J_{\frac{5}{4}}(x^2) + c_2 J_{-\frac{5}{4}}(x^2) \right]$

Q.18 Define the generating function for $J_n(x)$.

Ans. The function $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$ is called generating function.

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

$J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$.

Q.19 Prove that $erf(-x) = -erfx$

Ans. $$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2 dt}$$

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2 dt}$$

$$= -\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2 dt} = -erfx$$

Q.20 Prove that $P_n(-1) = (-1)^n$.

Ans. We know that $P_n(-x) = (-1)^n P_n(x)$

Put x = 1,

$$P_n(-1) = (-1)^n P_n(1)$$

$$= (-1)^n \qquad [\text{As } P_n(1) = 1]$$

Q.21 Prove that $P_n'(1) = \frac{n(n+1)}{2}$

Ans. We know Legendre's differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

$y=P_n(x)$ is the solution of above equation.

$$\left(1-x^2\right)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0$$

Put $x = 1$,

$$-2P_n'(1)+n(n+1)P_n(1)=0 \qquad [P_n(1)=1]$$

$$P_n'(1)=\frac{n(n+1)}{2}$$

Q.22 Prove that $4J_0'''(x)+3J_0'(x)+J_3(x)=0$.

Ans. We know $J_0' = -J_1$

Differentiation gives $J_0'' = -J_1' = -\frac{1}{2}(J_0 - J_2)$

Differentiation gives $J_0''' = -\frac{1}{2}(J_0' - J_2')$

$$= -\frac{1}{2}J_0' + \frac{1}{4}(J_1 - J_3)$$

$$= -\frac{3}{4}J_0' - \frac{1}{4}J_3$$

$$4J_0'''(x)+3J_0'(x)+J_3(x)=0$$

Q.23 Prove that $J_{n+3} + J_{n+5} = \frac{2}{x}(n+4)J_{n+4}$.

Ans. As we know, $2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x))$

Replace n by $n + 4$, we get:

$$\frac{2}{x}(n+4)J_{n+4} = J_{n+3} + J_{n+5}$$

Q.24 Show that $erf_c 0 = 1$.

Ans. As the relation $erf_c(x) + erf(x) = 1$

As $erf\, 0 = 0$

$$erf_c 0 = 1$$

Q.25 Solve the equation in terms of Bessel equation.

$$xy'' - 3y' + xy = 0.$$

Ans. $1 - 2\alpha = -3$, $ß^2 r^2 = 1$, $2r = 2$, $\alpha^2 - n^2 r^2 = 0$

$\alpha = 2$, $ß = r = 1$, $n = 2$

Hence the solution is

$$y = x^2[c_1 J_2(x) + c_2 Y_2(x)].$$

Q.26 Recurrence Relations of BESSEL function

1. Prove that $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

   Proof: Since

$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$

$$x^n J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2n+2k}}{2^{n+2k} k!\, \Gamma(n+k+1)}$$

$$\frac{d}{dx}[x^n J_n(x)] = \sum_{k=0}^{\infty} \frac{(-1)^k\,(2n+2k) x^{2n+2k-1}}{2^{n+2k} k!\, \Gamma(n+k+1)} = x^n \sum_{k=0}^{\infty} \frac{(-1)^k\,(n+k) x^{n+2k-1}}{2^{n+2k-1} k!\,(n+k)\Gamma(n+k)}$$

$$= x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k)} \left(\frac{x}{2}\right)^{n+2k-1} = x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n-1+k+1)} \left(\frac{x}{2}\right)^{n+2k-1}$$

$$= x^n J_{n-1}(x)$$

II) Prove that $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

Since $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$

$$x^{-n} J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k}}{2^{n+2k} k!\, \Gamma(n+k+1)}$$

$$\frac{d}{dx}[x^{-n}J_n(x)] = \sum_{k=0}^{\infty} \frac{(-1)^k\,(2k)x^{2k-1}}{2^{n+2k}k(k-1)!\,\Gamma(n+k+1)}$$

$$= x^{-n}\sum_{k=0}^{\infty} \frac{(-1)^{k-1}\,x^{n+2k-1}}{2^{n+3k-1}(k-1)!\,(n+k)\Gamma(n+k+1)}$$

$$= -x^{-n}\sum_{k=0}^{\infty} \frac{(-1)^r\,x^{n+2k+1}}{2^{n+2r+1}r!\,\Gamma(n+r+2)} \qquad where\, r = k-1$$

$$= -x^{-n}\sum_{k=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+1+r+1)} \left(\frac{x}{2}\right)^{n+1+2r} = -x^{-n}J_{n+1}(x)$$

III) Prove that $J_n{}'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)$

Proof:

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$=> \qquad x^n J_n{}'(x) + nx^{n-1}J_n(x) = x^n J_{n-1}(x)$$

Dividing by x^n, we have $J_n{}'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)$____________________(1)

IV) Prove that $J_n{}'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x)$

Proof: $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

$$=> \qquad x^{-n}J_n{}'(x) - nx^{n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

Multiplying by x^n , we have

$J_n{}'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x)$_ ______________________________________(2)

V) Prove that $2J_n{}'(x) = J_{n-1}(x) - J_{n+1}(x)$

Adding (1) and (2)

We have $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ ________________________________(3)

VI) Prove that $\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

Proof: Subtracting (2) from (1)

We get $\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

Q.27 Recurrence Relations for $P_n(x)$

i) $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

ii) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

iii) $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

iv) $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$

v) $(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$

vi) $(1-x^2)P_n'(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$

Proof: $(1-2xz+z^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$ (1)

Differentiating (1) partially w.r.t h, we get

$$\frac{-1}{2}(1-2xh+h^2)^{\frac{-3}{2}}(2h-2x) = \sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

$$(x-h)(1-2xh+h^2)^{\frac{-1}{2}} = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

$$(x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1} P_n(x)$$

Equating coefficients of h^{n-1} on both sides,

$$xP_{n-1}(x) - P_{n-2}(x) = nP_n(x) - 2x(n-1)P_{n-1}(x) + (n-2)P_{n-2}(x)$$

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

Replacing n by n+1, we get

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

ii) we know that : $(1-2xh+h^2)^{\frac{-1}{2}} = \sum_{n=0}^{\infty} h^n\, P_n(x)$

Differentiating (1) partially w.r.t h, we get

$$\frac{-1}{2}(1-2xh+h^2)^{\frac{-3}{2}}(2h-2x) = \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$$

$$(x-h)(1-2xh+h^2)^{\frac{-1}{2}} = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$$

$$(x-h)\sum_{n=0}^{\infty} h^n\, P_n(x) = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$$

Differentiating both sides w.r.t x

$$\frac{-1}{2}(1-2xh+h^2)^{\frac{-3}{2}}(-2h) = \sum_{n=0}^{\infty} h^n\, P'_n(x)$$

$$(x-h)(1-2xh+h^2)^{\frac{-3}{2}} = (x-h)\sum_{n=0}^{\infty} h^{n-1})\, P'_n(x)$$

Comparing the coefficient of h^{n-1} on both sides, we get

$$nP_n(x) = xP_n{}'(x) - P_{n-1}{}'(x)$$

iii) From recurrence relation (i) $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

Differentiating w.r.t.x, we get

$(2n+1)[xP_n{}'(x) + P_n(x)] = (n+1)P'_{n+1}(x) + nP'_{n-1}(x)$ …………………(1)

$$xP_n{}'(x) = nP_n(x) + P'_{n-1}(x)$$

From (1) $(2n+1)[nP_n(x) + P'_{n-1}(x) + P_n(x)] = (n+1)P'_{n+1}(x) + nP'_{n-1}(x)$

Hence $(2n+1)P_n(x) = P_{n+1}{}'(x) - P_{n-1}{}'(x)$

(iv) From recurrence relation (iii) $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

From recurrence relation (ii) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

$$(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$$

V) From recurrence relation (iv) $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$……………..1

From recurrence relation (ii) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$……………………2

Multiplying (2) by x subtracting from (1), we have

$$(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$$

(vi) From recurrence relation (i) $(\overline{n+1} + \bar{n})xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

$$(n+1)xP_n(x) + nxP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

$$(n+1)[xP_n(x) - P_{n+1}(x)] = (1-x^2)P_n'(x)$$

$\therefore (1-x^2)P_n'(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$

Q.28 Show that $x^4 = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$.

Ans. $P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$

$P_2(x) = \frac{1}{2}[3x^2 - 1]$

$P_0(x) = 1$

Hence $\frac{1}{35}\left[8\frac{1}{8}(35x^4 - 30x^2 + 3) + 20\frac{1}{2}(3x^2 - 1) + 7\right] = x^4$

Q.29 Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of Legendre's polynomials.

Ans. $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$; $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$; $x = P_1(x)$; $P_0(x) = 1$

$$x^3 - 5x^2 + x + 2 = \left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - 5\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + P_1(x) + 2P_0(x)$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x)$$